Local return rates in Sturmian subshifts

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The local return rates have been introduced by Hirata, Saussol and Vaienti [7] as a tool for the study of the asymptotic distribution of the return times to cylinders. We give formulas for these rates in Sturmian subshifts.

1 Introduction

The lower and upper local return rates have been introduced by Hirata, Saussol and Vaienti in [7] as a tool for the study of the asympotic distribution of the return times to cylinders in a class of non-uniformly hyperbolic dynamical systems. They are functions $\underline{R}_\xi, \overline{R}_\xi : X \longrightarrow [0,\infty]$ defined for an arbitrary topological dynamical system (X,F) and a finite partition ξ of X. For a subshift $\Sigma \subseteq A^{\mathbb{N}}$ and the canonical partition $\{[a] \mid a \in A\}$ we can reformulate the definition as

$$\begin{aligned}\underline{R}(x) &= \liminf_{n\to\infty} \frac{\tau\left([x(n)]\right)}{n}\\ \overline{R}(x) &= \limsup_{n\to\infty} \frac{\tau\left([x(n)]\right)}{n}.\end{aligned}$$

Here $x(n) = x_0x_1\dots x_{n-1}$ is a prefix of $x \in \Sigma$ of length n, $[x(n)]$ is its cylinder and $\tau([x(n)])$ is the Poincaré return time of $[x(n)]$.

For an arbitrary dynamical system (X,F) the functions $\underline{R}_\xi, \overline{R}_\xi$ are subinvariant, i.e., $\underline{R}_\xi \circ F \le \underline{R}_\xi$ and $\overline{R}_\xi \circ F \le \overline{R}_\xi$. Moreover, if μ is an F-invariant Borel probability measure and ξ is a measurable partition of X, then $\underline{R}$ and $\overline{R}$ are invariant allmost everywhere. In particular, if (X,F,μ) is ergodic, then by the Birkhoff ergodic theorem there exist constants $\mathbf{r}_0, \mathbf{r}_1 \in [0,\infty]$ such that for almost all $x \in X$, $\underline{R}_\xi(x) = \mathbf{r}_0$ and $\overline{R}_\xi(x) = \mathbf{r}_1$.

The ergodic case has been treated in several more papers. Saussol et al [9](see also [1]) show that if the entropy of μ is positive, then $\mathbf{r}_0 \ge 1$. Cassaigne et al [2] show that this inequality is not satisfied for systems with zero entropy. In particular for the Fibonacci shift obtained from the golden angle rotation, the lower local rate assumes the value $\mathbf{r}_0 = \frac{3-\sqrt{5}}{2} < 1$. Afraimovich et al [1] show that $\mathbf{r}_0 = 0$ for some rotations of the circle whose parameter has unbounded continued fraction expansion. It follows that the same result holds

for the corresponding Sturmian subshift. Kůrka [8] treats the case of substitutive subshifts and obtains a formula for $\mathbf{r}_0$ and $\mathbf{r}_1$. In this case both $\mathbf{r}_0$ and $\mathbf{r}_1$ are positive and finite.

In this paper we will discuss completely the situation in the Sturmian shifts. One can easy check that the result of Afraimovich et al considered for corresponding Sturmian shifts and the result of Cassaigne et al for Fibonacci shift follows immediately. We give formulas for $\mathbf{r}_0$ and $\mathbf{r}_1$ in terms of the convergents q_k obtained from the continued fraction expansion of the parameter $\alpha = [0, a_1, a_2, \ldots]$. If a_k are bounded, then $\mathbf{r}_0$ and $\mathbf{r}_1$ are positive and finite. If a_k are unbounded, then $\mathbf{r}_0 = 0$ and $\mathbf{r}_1 = \infty$. This result, that $\mathbf{r}_0 = 0$ iff $\mathbf{r}_1 = \infty$ iff the continued fraction expansion is unbounded, has been obtained by a different technique by Chazottes and Durand in [3]

2 Sturmian shifts

A dynamical system is a pair (X, F), where X is a compact metric space and F is a continuous function from X to X. The Poincaré return time of a subset $M \subseteq X$ is

$$\tau(M) = \min\{k > 0 \mid F^k(M) \cap M \neq \emptyset\}.$$

Let A be a finite alphabet, and $A^{\mathbb{N}}$ the space of all infinite sequences of letters from A with the product topology. The set A^* consists of all words (finite sequences) over A. For a word $u = u_0 u_2 \ldots u_{n-1} \in A^*$, denote by $|u| = n$ its length. The set A^n consists of all words of length n. The shift map $\sigma : A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ is defined by $\sigma_i(x) = x_{i+1}$.

A shift is any subsystem (Σ, σ) of $(A^{\mathbb{N}}, \sigma)$, where $\Sigma \subseteq A^{\mathbb{N}}$ is nonempty, closed and σ-invariant. For a shift Σ and for a word $u = u_0 u_1 \ldots u_{n-1} \in A^*$ we denote by $[u] = \{x \in \Sigma \mid \forall i < n : x_i = u_i\}$ the cylinder of u. The language of a shift is the set of words which have nonempty cylinders, i.e., $\mathcal{L}(\Sigma) = \{u \in A^* \mid [u] \neq \emptyset\}$. The set $\mathcal{L}^n(\Sigma)$ consists of all words of the language of length n. If we denote by $x(n) = x_0 x_1 \ldots x_{n-1}$ the prefix of $x \in \Sigma$ of length n, then $\mathcal{L}^n(\Sigma) = \{x(n) \mid x \in \Sigma\}$.

A Sturmian shift is a coding of an irrational rotation of the unit circle (Hedlund and Morse [6]). This is a dynamical system $(\mathbb{T}, F_\alpha)$, where $\mathbb{T} = [0, 1[$ is the circle with the metric $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$ and $F_\alpha(x) = x + \alpha \mod 1$, where $\alpha \in \mathbb{R}$. We consider only irrational angles from the open interval $\alpha \in]0, 1[$.

There is the canonical partition $\mathcal{I} = \{I_0, I_1\}$ of $\mathbb{T}$, where $I_0 = [0, 1 - \alpha[$ and $I_1 = [1 - \alpha, 1[$. For $u \in \mathbf{2}^*$, set

$$I_u = \bigcap_{k=0}^{|u|-1} F_\alpha^{-k}(I_{u_k}).$$

Any I_u is either a semiopen interval or the empty set. The associated Sturmian shift (Σ_α, σ) is defined by its language $\mathcal{L}(\Sigma_\alpha) = \{u \in \mathbf{2}^* \mid I_u \neq \emptyset\}$. In other

words,

$$\Sigma_\alpha = \{x \in \mathbf{2}^{\mathbb{N}} \mid \forall n \in \mathbb{N}, I_{x(n)} \neq \emptyset\}.$$

If $\alpha \in]0,1[$ is irrational, both the rotation $(\mathbb{T}, F_\alpha)$ and the Sturmian shift (Σ_α, σ), are minimal and uniquely ergodic. Moreover, if $u \in \mathcal{L}(\Sigma_\alpha)$, then

$$\mu([u]) = |I_u|, \quad \tau([u]) = \tau(I_u),$$

where $|I_u|$ is the length of the interval I_u. It follows that the local return rates can be computed from the return times of intervals.

$$\begin{aligned} \underline{R}(x) &= \liminf_{n\to\infty} \frac{\tau(I_{x(n)})}{n} \\ \overline{R}(x) &= \limsup_{n\to\infty} \frac{\tau(I_{x(n)})}{n}. \end{aligned}$$

The description of the intervals I_u is obtained from the continued fraction expansion of α. There exists a unique sequence $\{a_k\}_{k=1}^\infty$ of positive integers such that

$$\alpha = [0, a_1, a_2, \ldots] = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}.$$

The convergents of α are the sequences $\{p_k\}_{k=-1}^\infty$, $\{q_k\}_{k=-1}^\infty$ defined by $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = 0$, $q_0 = 1$ and

$$q_{k+1} = a_{k+1} q_k + q_{k-1}, \quad p_{k+1} = a_{k+1} p_k + p_{k-1}.$$

By the Klein theorem (see Hardy and Wright [5]), the closest returns of the iterates $F_\alpha^n(0)$ to zero happen at times q_k. We have $d(0, F^{q_k}(0)) = \eta_k = (-1)^k (q_k \alpha - p_k)$ and for $q_k < n < q_{k+1}$, $d(0, F^n(0)) > \eta_k$. In particular $\eta_{-1} = 1$, $\eta_0 = \alpha$ and

$$\eta_{k+1} = a_{k+1} \eta_k - \eta_{k-1}.$$

The sequence $\{\eta_k\}_{k=-1}^\infty$ is positive, decreasing and converges to zero. It follows that if $I = [a, b[$ is a semiopen interval, then

$$\eta_{k+1} < |I| \leq \eta_k \;\Rightarrow\; \tau(I) = q_{k+1}.$$

The return times of intervals from $\mathcal{I}^n$ are therefore convergents q_k. We determine times when the return times jump from some q_k to a higher q_{k+1} (or q_{k+2}) and obtain a formula for the local return rates.

3 Jumps of the return time

Proposition 1. *For $x \in \Sigma_\alpha$, $k \geq -1$, define the k-th jump of the return time as*

$$r_k(x) = \min\{n \in \mathbb{N} \mid \tau(I_{x(n)}) \geq q_{k+1}\}.$$

Then $r_{-1}(x) = 0$ *and the following equalities hold for* $x \in \Sigma_\alpha$.

$$\begin{aligned}
\underline{R}(x) &= \liminf_{k\to\infty} \frac{q_k}{r_k(x)} = 1/\limsup_{k\to\infty} \frac{r_k(x)}{q_k} \\
\overline{R}(x) &= \limsup_{k\to\infty} \frac{q_{k+1}}{r_k(x)} = 1/\liminf_{k\to\infty} \frac{r_k(x)}{q_{k+1}}
\end{aligned}$$

Proof. For $x \in \Sigma_\alpha$, denote $S = \{k \in \mathbb{N} \mid r_{k-1}(x) < r_k(x)\}$. The set is infinite and we can order it into increasing sequence $\{k_i\}_{i=0}^\infty$. If $r_{k_i}(x) \le n < r_{k_{i+1}}(x)$, then $\tau(I_{x(n)}) = q_{k_{i+1}}$ and if $k_i < k < k_{i+1}$, then $\frac{q_k}{r_k(x)} \ge \frac{q_{k_i}}{r_{k_i}(x)}$, $\frac{q_k}{r_{k-1}(x)} \le \frac{q_{k_{i+1}}}{r_{k_i}(x)}$. Thus

$$\begin{aligned}
\underline{R}(x) &= \liminf_{n\to\infty} \frac{\tau(I_{x(n)})}{n} = \liminf_{i\to\infty} \left(\min_{r_{k_i}(x)\le n<r_{k_{i+1}}(x)} \frac{\tau(I_{x(n)})}{n} \right) = \\
&= \liminf_{i\to\infty} \frac{q_{k_{i+1}}}{r_{k_{i+1}}(x)-1} = \liminf_{i\to\infty} \frac{q_{k_i}}{r_{k_i}(x)} = \liminf_{k\to\infty} \frac{q_k}{r_k(x)}. \\
\overline{R}(x) &= \limsup_{i\to\infty} \left(\max_{r_{k_i}(x)\le n<r_{k_{i+1}}(x)} \frac{\tau(I_{x(n)})}{n} \right) = \limsup_{k\to\infty} \frac{q_{k+1}}{r_k(x)}.
\end{aligned}$$

□

To compute the jumps of the return time, we construct another symbolic description of Sturmian shifts. The partition $\mathcal{I}^n = \{I_u \mid u \in \mathcal{L}^n(\Sigma_\alpha)\}$ consists of semiopen intervals on the unit circle divided by cut points

$$\mathrm{Cut}(n) = \{\langle i\rangle \mid i = 0, 1, \ldots, n\},$$

where $\langle i\rangle = F_\alpha^{-i}(0) = (-i\alpha) \mod 1$. The structure of $\mathcal{I}^n$ is described by the Three length theorem (Sós [10]) which says that $\mathcal{I}^n$ contains intervals of at most three lengths. For some n, however $\mathcal{I}^n$ contains only intervals of two lengths. This happens in particular at times $n = q_k - 1$, when the intervals of $\mathcal{I}^n$ have lengths η_{k-1} and $\eta_{k-1} + \eta_k$. To describe the partitions $\mathcal{I}^{q_k-1}$ we consider a new symbolic space X_α which consists of paths in the infinite graph in Figure 1. It looks like Bratelli diagram([4]), but the dynamics on X_α is far more complicated. The main reason for introducing the space X_α is to obtain a simple formula for $r_k(x)$ in Proposition 3.

Definition 1. *For an irrational* $\alpha = [0, a_1, a_2, \ldots]$ *set*

$$\begin{aligned}
X_\alpha &= \left\{ x \in \prod_{k=1}^\infty \{0,1,\ldots,a_k\} \mid x_1 \ne 0, \quad (x_{k+1} = 0 \Rightarrow x_k = a_k) \right\} \\
\mathcal{L}^n(X_\alpha) &= \left\{ u \in \prod_{k=1}^n \{0,1,\ldots,a_k\} \mid u_1 \ne 0, \quad (u_{k+1} = 0 \Rightarrow u_k = a_k) \right\} \\
\mathcal{L}(X_\alpha) &= \bigcup_{n\ge 1} \mathcal{L}^n(X_\alpha).
\end{aligned}$$

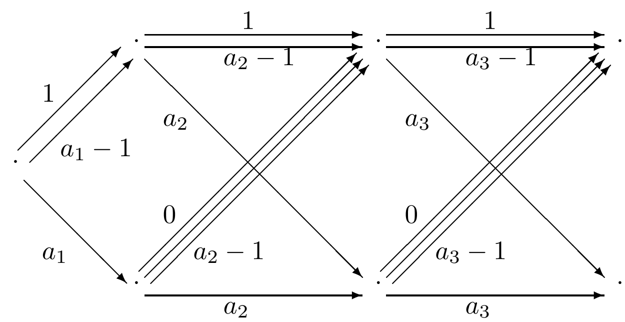

Figure 1: The symbolic space X_α

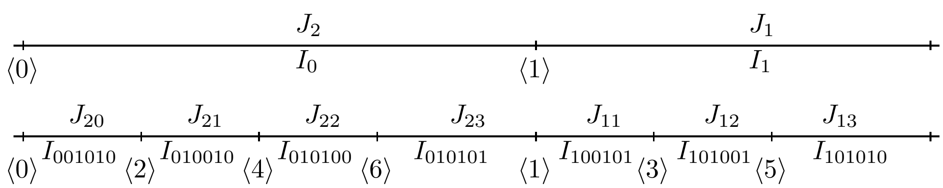

Figure 2: Partitions of the circle

We construct a system of intervals $\{J_u \mid u \in \mathcal{L}(X_\alpha)\}$. *If* $1 \leq u_1 \leq a_1$ *set*

$$J_{u_1} = \begin{cases} [\langle u_1 \rangle, \langle u_1 - 1 \rangle[& \text{if} \quad u_1 < a_1 \\ [\langle 0 \rangle, \langle a_1 - 1 \rangle[& \text{if} \quad u_1 = a_1 \end{cases}$$

If $u \in \mathcal{L}^k(X_\alpha)$, $k > 1$, $J_{u(k-1)} = (-1)^{k-2}[\langle a \rangle, \langle b \rangle[$, *and if* $u_{k-1} < a_{k-1}$ *set*

$$J_u = \begin{cases} (-1)^{k-1}[\langle u_k q_{k-1} + a \rangle, \langle (u_k - 1) q_{k-1} + a \rangle[& \text{if} \quad 1 \leq u_k \leq a_k - 1 \\ (-1)^{k-1}[\langle b \rangle, \langle (a_k - 1) q_{k-1} + a \rangle[& \text{if} \quad u_k = a_k \end{cases}$$

If $u_{k-1} = a_{k-1}$ *set*

$$J_u = \begin{cases} (-1)^{k-1}[\langle (u_k + 1) q_{k-1} + a \rangle, \langle u_k q_{k-1} + a \rangle[& \text{if} \quad 0 \leq u_k \leq a_k - 1 \\ (-1)^{k-1}[\langle b \rangle, \langle u_k q_{k-1} + a \rangle[& \text{if} \quad u_k = a_k \end{cases}$$

Here $(-1)[b, a[= [a, b[$, where $0 \leq a < b < 1$, is a semiopen interval of the circle. We identify also $[a, 0[= [a, 1[= (-1)[0, a[$. If $x \in X_\alpha$, we denote by $x(n) = x_1 \ldots x_n$ the prefix of x of length n. In Figure 2 we can see the partitions of the circle for $\alpha = [0, 2, 3, \ldots]$.

Proposition 2. *If* $u \in \mathcal{L}^k(X_\alpha)$, $k \geq 1$, *then*

$$|J_u| = \begin{cases} \eta_{k-1} & \text{if} \quad u_k < a_k \\ \eta_{k-1} + \eta_k & \text{if} \quad u_k = a_k \end{cases}$$

and

$$\begin{aligned}\mathcal{I}^{q_k-1} &= \{J_u \mid u \in \mathcal{L}^k(X_\alpha)\} \\ &= \{(-1)^{k-1}[\langle i+q_{k-1}\rangle, \langle i\rangle[\mid i=0,1,\dots,q_k-q_{k-1}-1\} \cup \\ &\quad\ \{(-1)^{k-1}[\langle i\rangle, \langle i+q_k-q_{k-1}\rangle[\mid i=0,1,\dots,q_{k-1}-1\}.\end{aligned}$$

Moreover, $\mathcal{J}_u = \{J_v \mid v \in \mathcal{L}^{k+1}(X_\alpha) | v(k) = u\}$ *is a partition of* J_u *and*

$$\mathcal{I}^{q_{k+1}-1} = \bigcup_{u \in \mathcal{L}^k(X_\alpha)} \mathcal{J}_u.$$

Proof. If $u \in \mathcal{L}^k(X_\alpha)$, $k \geq 1$, $u_k < a_k$, then J_u is an image of $(-1)^{k-1}[\langle q_{k-1}\rangle, 0[$ in a rotation. By the Klein theorem, $|I_u| = \eta_{k-1}$.

We have $1 > \langle 1\rangle > \langle 2\rangle > \cdots \langle a_1-1\rangle > 0$, so $\{J_{u_1} \mid 1 \leq u_1 \leq a_1\} = \mathcal{I}^{q_1-1}$ and $|J_{a_1}| = 1-(a_1-1)\eta_0 = \eta_0+\eta_1$. Assume that the first part of the proposition holds for $k \geq 1$. Let $u \in \mathcal{L}^k(X_\alpha)$. Intervals from $M = \{J_{uj} \mid j < a_{k+1}\}$ coincide and we have proved that its length is η_k. Denote $J = \bigcup M$. If $u_k < a_k$ then $|J| = (a_{k+1}-1)\eta_k$ and if $u_k = a_k$ then $|J| = a_{k+1}\eta_k$. In both cases, $|J| < |J_u|$, $J_{ua_{k+1}} = J_u - J$ and $|J_{ua_{k+1}}| = \eta_{k-1} - (a_{k+1}-1)\eta_k = \eta_k + \eta_{k+1}$. Thus $\mathcal{J}_u$ is a partition of J_u. Because $\{J_u \mid u \in \mathcal{L}^k(X_\alpha)\}$ is a partition of $\mathbb{T}$, then also

$$\mathcal{J} = \{J_v \mid v \in \mathcal{L}^{k+1}(X_\alpha)\} = \bigcup_{u \in \mathcal{L}^k(X_\alpha)} \mathcal{J}_u$$

is. It is not difficult to prove that the endpoints of intervals from $\mathcal{J}$ belong to $\{\langle i\rangle \mid 0 \leq i \leq q_{k+1}-1\}$. The partitions $\mathcal{J}$ and $\mathcal{I}^{q_{k+1}-1}$ contain intervals of two lengths η_k and $\eta_k + \eta_{k+1}$, hence $\mathcal{J} = \mathcal{I}^{q_{k+1}-1}$. For the partition

$$\begin{aligned}\mathcal{J}' &= \{(-1)^k[\langle i+q_k\rangle, \langle i\rangle[\mid i=0,1,\dots,q_{k+1}-q_k-1\} \cup \\ &\quad\ \{(-1)^k[\langle i\rangle, \langle i+q_{k+1}-q_k\rangle[\mid i=0,1,\dots,q_k-1\}\end{aligned}$$

we prove the equality $\mathcal{I}^{q_{k+1}-1} = \mathcal{J}'$ similarly. □

For each $k \geq 1$ we have thus a one-to-one map $\gamma_k : \mathcal{L}^{q_k-1}(\Sigma_\alpha) \to \mathcal{L}^k(X_\alpha)$ given by $J_{\gamma_k(u)} = I_u$. For the corresponding symbolic spaces we get a homeomorphism $\gamma : \Sigma_\alpha \to X_\alpha$ given by $\gamma(x)(k) = \gamma_k(x(q_k-1))$. The local return rates, as well as the functions of the return jumps are carried over to the space X_α. By the abuse of notation we keep for them the same symbols $\underline{R}, \overline{R} : X_\alpha \to [0,\infty]$, $r_k : X_\alpha \to \mathbb{N}$. We now obtain a recursive formula for r_k.

Proposition 3. *For* $x \in X_\alpha$ *we have* $r_{-1}(x) = 0$ *and*

$$r_k(x) = x_{k+1}q_k + r_{k-1}(x) = \sum_{j=0}^{k} x_{j+1}q_j.$$

Proof. Assume $y \in \Sigma_\alpha$, $x = \gamma(y) \in X_\alpha$ and $k \geq 0$. We show first that if $J_{x(k)} = (-1)^{k-1}[\langle a\rangle, \langle b\rangle[$, then $r_{k-1}(x) = b+q_{k-1}$. If $x_k < a_k$, then $J_{x(k)} = I_{y(a)}$. Since $I_{y(a-1)} \neq I_{y(a)}$, $|I_{y(a-1)}| > \eta_{k-1}$ and $r_{k-1}(x) = a = b+q_{k-1}$. Let $x_k = a_k$. Since the form of partition $\{J_u \mid u \in \mathcal{L}^{k+1}(X_\alpha), u(k) = x(k)\}$ of $J_{x(k)}$ we get $J_{x(k)} = I_1 \cup I_2$ where

$$I_1 = (-1)^{k-1}[\langle a\rangle, \langle a+q_k\rangle[, \quad I_2 = (-1)^{k-1}[\langle a+q_k\rangle, \langle a+q_k-q_{k-1}\rangle[,$$

$I_1, I_2 \in \mathcal{I}^{a+q_k}$, $|I_1| = \eta_k$, $|I_2| = \eta_{k-1}$ and $I_{y(a+q_k-1)} = J_{x(k)}$ and either $I_{y(a+q_k)} = I_1$ or $I_{y(a+q_k)} = I_2$. Hence $|I_{y(a+q_k)}| \leq \eta_{k-1}$, $|I_{y(a+q_k-1)}| > \eta_{k-1}$ and $r_{k-1}(x) = a + q_k = b + q_{k-1}$.

Assume now that $J_{x(k+1)} = (-1)^k[\langle c\rangle, \langle d\rangle[$, so $r_k(x) = d + q_k$. Put j=1 if $x_k = a_k$, j=0 otherwise. It follows $d = a + (x_{k+1}) - 1)q_k + jq_k$ and $a = b + q_{k-1} - jq_k$. Thus

$$\begin{aligned} r_k(x) - r_{k-1}(x) &= (d+q_k) - (b+q_{k-1}) = (a + x_{j+1}q_k + jq_k) - (a + jq_k) \\ &= x_{k+1}q_k. \end{aligned}$$

□

Proposition 4. *For every $x \in X_\alpha$ we have $q_k \leq r_k(x) \leq q_{k+1} + q_k - 1$.*

Proof. Clearly $q_{-1} = 0 = r_{-1}(x) = 0 = q_0 + q_{-1} - 1$, $q_1 = 1 \leq r_1(x) \leq a_1 = q_1 + q_0 - 1$. Assume that the statement holds for all integers less than k. Then

$$r_k(x) = x_{k+1}q_k + r_{k-1}(x) \leq a_{k+1}q_k + q_k + q_{k-1} - 1 = q_k + q_{k+1} - 1$$

If $x_{k+1} \geq 1$, then $r_k(x) = x_{k+1}q_k + r_{k-1}(x) \geq q_k$. If $x_{k+1} = 0$, then $x_k = a_k$ and

$$r_k(x) = r_{k-1}(x) = a_k q_{k-1} + r_{k-2}(x) \geq a_k q_{k-1} + q_{k-2} = q_k.$$

□

Proposition 5. *Define the points $b, c, d \in X_\alpha$ by*

$$b = (a_1, a_2, a_3, \ldots), \quad c = (1, a_2, 0, a_4, 0, a_6, \ldots), \quad d = (a_1, 0, a_3, 0, a_5, \ldots)$$

Then

$$\begin{aligned} \min \underline{R} &= \underline{R}(b) = \liminf_{k\to\infty} \frac{q_k}{q_{k+1} + q_k - 1} = \mathbf{r}_0 \\ \min \overline{R} &= \overline{R}(b) = \limsup_{k\to\infty} \frac{q_{k+1}}{q_{k+1} + q_k - 1} \\ \max \overline{R} &= \max(\overline{R}(c), \overline{R}(d)) = \limsup_{k\to\infty} \frac{q_{k+1}}{q_k} = \mathbf{r}_1. \end{aligned}$$

Proof. It is easy to see that for $k \in \mathbb{N}$,

$$\begin{aligned}
r_k(b) &= \sum_{j=0}^{k} a_{j+1} q_j = q_{k+1} + q_k - 1 \\
r_{2k-1}(c) = r_{2k}(c) &= 1 + \sum_{j=1}^{k} a_{2j} q_{2j-1} = q_{2k} \\
r_{2k}(d) = r_{2k+1}(d) &= \sum_{j=0}^{k} a_{2j+1} q_{2j} = q_{2k+1}
\end{aligned}$$

By Proposition 4 we obtain the bounds for the limits in the right hand sides. The following formulas complete the proof.

$$\begin{aligned}
\underline{R}(b) &= \liminf_{k\to\infty} \frac{q_k}{r_k(b)} = \mathbf{r}_0 \\
\overline{R}(b) &= \limsup_{k\to\infty} \frac{q_{k+1}}{r_k(b)} = \limsup_{k\to\infty} \frac{q_{k+1}}{q_{k+1} + q_k - 1} \\
\max(\overline{R}(c), \overline{R}(d)) &\geq \max\left(\limsup_{k\to\infty} \frac{q_{2k+1}}{r_{2k}(c)}, \limsup_{k\to\infty} \frac{q_{(2k-1)+1}}{r_{2k-1}(d)} \right) \\
&\geq \max\left(\limsup_{k\to\infty} \frac{q_{2k+1}}{q_{2k}}, \limsup_{k\to\infty} \frac{q_{2k}}{q_{2k-1}} \right) \\
&\geq \limsup_{k\to\infty} \frac{q_{k+1}}{q_k} = \mathbf{r}_1 .
\end{aligned}$$

□

We have not been able to obtain a formula for $\min \overline{R}$. Our results, however are sufficient to get formulas for $\mathbf{r}_0$ and $\mathbf{r}_1$. Now, put some bounds for the values $\mathbf{r}_0$ and $\mathbf{r}_1$.

Proposition 6. *Let $\alpha = [0, a_1, a_2, ..]$ be irrational, $M = \limsup a_k$, $\gamma = \frac{\sqrt{5}+1}{2}$. If the continued fraction expansion of α is unbounded, then $\mathbf{r}_0 = 0$ and $\mathbf{r}_1 = \infty$. Otherwise, $\mathbf{r}_1 = \frac{1}{\mathbf{r}_0} - 1$ and*

$$\frac{1}{M+2} \leq \mathbf{r}_0 \leq \gamma^{-2} < \gamma \leq \mathbf{r}_1 \leq M + 1$$

Moreover, $\mathbf{r}_1 = \gamma$(resp. $\mathbf{r}_0 = \gamma^{-2}$) if and only if $M = 1$.

Proof. Let $\alpha = [0, a_1, a_2, ..]$ be irrational, $M = limsup\ a_k$, $\gamma = \frac{\sqrt{5}+1}{2}$. Denote $B_k = \frac{q_{k+1}}{q_k}$. Then $a_k \leq B_k \leq a_{k+1} + 1$ and $\mathbf{r}_1 = \limsup B_k$,

$$\mathbf{r}_0 = \liminf \frac{1}{B_k + 1 - \frac{1}{q_k}} = \liminf \frac{1}{B_k + 1}$$

If the continued fraction expansion of α is unbounded, then also $\{B_k\}_{k=0}^{\infty}$ is. Hence $\mathbf{r}_0 = 0$ and $\mathbf{r}_1 = \infty$.

Let $M \in \mathbb{N}$. Then $M \leq \mathbf{r}_0 \leq M+1$. If $M \geq 2$, then $\gamma < M \leq \mathbf{r}_0$. If $M = 1$, then there exists $n_0 \in \mathbb{N}$, such that for every $n > n_0$, $a_n = 1$. Hence

$$\limsup B_k = 1 + \frac{1}{\liminf B_k} \quad , \quad \liminf B_k = 1 + \frac{1}{\limsup B_k}$$

It implies that $\mathbf{r}_0 = 1 + \frac{1}{1+\frac{1}{\mathbf{r}_0}}$. This equality have just one positive solution $\mathbf{r}_0 = \gamma$. All properties of $\mathbf{r}_1$ is given by the equality $\mathbf{r}_0 = \frac{1}{\mathbf{r}_1+1}$. $\square$

4 The measure

We are going to show that the constants $\mathbf{r}_0$ and $\mathbf{r}_1$ are assumed by $\underline{R}$ and $\overline{R}$ almost everywhere. The unique invariant measure μ on Σ_α is carried over to the space X_α using the length of associated intervals. If $u \in \mathcal{L}^k(X_\alpha)$, then the measure of the cylinder of u is $\mu\{x \in X_\alpha | x(k) = u\} = |J_u|$. Define the projections $W_k : X_\alpha \to \{0, 1, \ldots, a_k\}$ by $W_k(x) = x_k$. Then W_k are random variables and $(W_k)_{k\geq 1}$ is a nonstationary Markov chain. Using Proposition 2 we get the transition probabilities.

$$\begin{array}{rcll}
\mu[W_1 = j] & = & \eta_0 = \alpha & \text{for} \quad 1 \leq j < a_1 \\
\mu[W_1 = j] & = & \eta_0 + \eta_1 & \text{for} \quad j = a_1 \\
\mu[W_{k+1} = j | W_k < a_k] & = & \dfrac{\eta_k}{\eta_{k-1}} & \text{for} \quad 1 \leq j < a_{k+1} \\
\mu[W_{k+1} = j | W_k < a_k] & = & \dfrac{\eta_k + \eta_{k+1}}{\eta_{k-1}} & \text{for} \quad j = a_{k+1} \\
\mu[W_{k+1} = j | W_k = a_k] & = & \dfrac{\eta_k}{\eta_{k-1} + \eta_k} & \text{for} \quad 0 \leq j < a_{k+1} \\
\mu[W_{k+1} = j | W_k = a_k] & = & \dfrac{\eta_k + \eta_{k+1}}{\eta_{k-1} + \eta_k} & \text{for} \quad j = a_{k+1}
\end{array}$$

Theorem 7. *If the continued fraction expansion is unbounded, then $\underline{R}(x) = 0$, $\overline{R}(x) = \infty$ almost everywhere.*

Proof. For every $x \in X_\alpha$ we have

$$x_{k+1} \leq \frac{x_{k+1} q_k + r_{k-1}(x)}{q_k} = \frac{r_k(x)}{q_k}$$

Given $m \geq 1$ then $C_m = \{k \geq 1 | a_k \geq m\}$ is an infinite set. Assume that $k+1 \in C_{2m+1}$. We have

$$\begin{array}{rcl}
\mu[W_{k+1} \leq m | W_k < a_k] & = & \dfrac{m\eta_k}{\eta_{k-1}} = \dfrac{m\eta_k}{a_{k+1}\eta_k + \eta_{k+1}} \leq \dfrac{m}{a_{k+1}} \leq \dfrac{1}{2} \\
\mu[W_{k+1} \leq m | W_k = a_k] & = & \dfrac{(m+1)\eta_k}{\eta_{k-1} + \eta_k} \leq \dfrac{m+1}{a_{k+1}+1} \leq \dfrac{1}{2}
\end{array}$$

It follows that $\mu[W_{k+1} \leq m | W_j = i] \leq \frac{1}{2}$ for any $j \leq k$ and any $i \in \{0, 1, \ldots, a_j\}$. Given $k_0 > 0$ let $k_0 < k_1 < \cdots k_n$ be a sequence of integers from C_{2m+1}. Then

$$\begin{aligned} \mu[W_{k_1} \leq m, \ldots, W_{k_n} \leq m] &= \mu[W_{k_1} \leq m] \cdot \mu[W_{k_2} \leq m | W_{k_1} \leq m] \cdots \\ &\quad \mu[W_{k_n} \leq m | W_{k_1} \leq m, \ldots, W_{k_{n-1}} \leq m] \\ &\leq 2^{-n+1} \end{aligned}$$

It follows

$$\mu \left\{ x \in X_\alpha \left| \frac{r_{k_i}(x)}{q_{k_i}} \leq m, 1 \leq i \leq n \right. \right\} \leq \mu\{x \in X_\alpha \mid x_{k_i} \leq m, 1 \leq i \leq n\} \leq 2^{-n+1}$$

so $\mu\{x \in X_\alpha | \underline{R}(x) > \frac{1}{m}\} = 0$ and $\underline{R}(x) = 0$ almost everywhere. We prove now the statement for $\overline{R}$. Given $\varepsilon \in]0, 1[$, let m be an integer with $1 - \varepsilon + \frac{4}{m} = \delta < 1$. Assume that $k + 1 \in C_m$ and let $x \in X_\alpha$ be such that $r_k(x)/q_{k+1} \geq \varepsilon$. Then

$$\begin{aligned} \varepsilon &\leq \frac{x_{k+1} q_k + r_{k-1}(x)}{a_{k+1} q_k + q_{k-1}} \leq \frac{x_{k+1} q_k + q_k + q_{k-1}}{a_{k+1} q_k} \leq \frac{x_{k+1} + 2}{a_{k+1}} \\ x_{k+1} &\geq \varepsilon a_{k+1} - 2 = \varepsilon_k \end{aligned}$$

The probability of this event is bounded away from one. For any $j \leq a_k$ we have

$$\begin{aligned} \mu[W_{k+1} \geq \varepsilon_{k+1} | W_k = j] &\leq \frac{((1-\varepsilon) a_{k+1} + 3)\eta_k + \eta_{k+1}}{\eta_{k-1}} \\ &\leq \frac{((1-\varepsilon) a_{k+1} + 4)\eta_k}{a_{k+1} \eta_k} \leq \delta \end{aligned}$$

It follows that $\mu[W_{k+1} \geq \varepsilon_{k+1} | W_j = i] \geq \delta$ whenever $j \leq k$ and $i \in \{0, \ldots, a_j\}$. Given $k_0 > 0$, let $k_0 < k_1 < k_2 < \cdots < k_n$ be an incresing sequence of indices from C_m. Then $\mu[W_{k_1} \geq \varepsilon_{k_1}, \ldots, W_{k_n} \geq \varepsilon_{k_n}] \leq \delta^n$. It follows

$$\mu \left\{ x \in X_\alpha \left| \frac{r_{k_i}(x)}{q_{k_{i+1}}} \geq \varepsilon, 1 \leq i \leq n \right. \right\} \leq \mu[x \in X_\alpha | x_{k_i} \geq \varepsilon_{k_i}, 1 \leq i \leq n] \leq \delta^n$$

so $\mu\{x \in X_\alpha : \overline{R}(x) < \frac{1}{\varepsilon}\} = 0$ and $\overline{R}(x) = \infty$ almost everywhere. □

Proposition 8. *If α have bounded coeficients in its continued fraction, then $\underline{R}(x) = \mathbf{r}_0$, $\overline{R}(x) = \mathbf{r}_1$ almost everywhere.*

Proof. Proposition 5 says that $\min \underline{R} = \underline{R}(b)$, where $b = (a_1, a_2, a_3, \ldots)$. We are going to prove that

$$\mu \left\{ x \in X_\alpha \left| \limsup_{k \to \infty} \frac{r_k(x)}{q_k} \leq \limsup_{k \to \infty} \frac{r_k(b)}{q_k} \right. \right\} = 1.$$

Fix $m \geq 1$. There exists an integer sequence $\{n_k\}_{k=0}^\infty$ such that $n_0 > m$, $n_k - n_{k-1} > m$, for $k \geq 1$ and

$$\limsup_{n \to \infty} \frac{r_n(e)}{q_n} = \lim_{k \to \infty} \frac{r_{n_k}(e)}{q_{n_k}}.$$

For $k \in \mathbb{N}$, set

$$D_k = \{x \in X_\alpha \mid x_{n_k} = b_{n_k}, x_{n_k-1} = b_{n_k-1}, \ldots, x_{n_k-m+1} = b_{n_k-m+1}\}.$$

and $D = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} D_k$. We show $\mu(D) = 1$. Let M be a bound for the continued fraction expansion, so $a_k \leq M$ for every k. Then for any $i \leq a_k$, $j \leq a_{k+1}$ we have

$$\mu[W_{k+1} = j | W_k = i] \geq \frac{\eta_k}{\eta_k + \eta_{k+1}} \geq \frac{1}{a_{k+1}+2} \geq \frac{1}{M+2}$$

It follows $\mu(X_\alpha \setminus D_k) \leq 1 - \frac{1}{(M+2)^n}$ and

$$\mu(D) = 1 - \mu\left(\bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (X_\alpha \setminus D_k)\right) = 1 - 0 = 1.$$

Given $x \in D$, there exists an increasing integer sequence $\{k_j\}_{j=1}^{\infty}$ such that $x \in D_{n_{k_j}}$. For each j, we have

$$\begin{aligned}
r_{n_{k_j}}(b) - r_{n_{k_j}}(x) &= \sum_{i=0}^{n_{k_j}} b_{i+1} q_i - \sum_{i=0}^{n_{k_j}} x_{i+1} q_i = \sum_{i=0}^{n_{k_j}-m} b_{i+1} q_i - \sum_{i=0}^{n_{k_j}-m} x_{i+1} q_i \\
&= r_{n_{k_j}-m}(b) - r_{n_{k_j}-m}(x) \leq q_{n_{k_j}-m+1} + q_{n_{k_j}-m} - 1 - q_{n_{k_j}-m} \\
&\leq q_{n_{k_j}-m+1}.
\end{aligned}$$

Since $q_{n+2} = a_{n+2} q_{n+1} + q_n \geq 2q_n$, we get

$$\frac{r_{n_{k_j}}(b)}{q_{n_{k_j}}} - \frac{r_{n_{k_j}}(x)}{q_{n_{k_j}}} \leq \frac{q_{n_{k_j}-m+1}}{q_{n_{k_j}}} \leq \frac{2^{-\lfloor \frac{m-1}{2} \rfloor} q_{n_{k_j}}}{q_{n_{k_j}}} = 2^{-\lfloor \frac{m-1}{2} \rfloor}.$$

and

$$\limsup_{k\to\infty} \frac{r_k(x)}{q_k} \geq \limsup_{j\to\infty} \frac{r_{k_j}(b)}{q_{k_j}} - 2^{-\lfloor \frac{m-1}{2} \rfloor} = \limsup_{k\to\infty} \frac{r_k(b)}{q_k} - 2^{-\lfloor \frac{m-1}{2} \rfloor}.$$

It follows

$$\mu\left\{x \in X_\alpha \,\middle|\, \limsup_{k\to\infty} \frac{r_k(x)}{q_k} \geq \limsup_{k\to\infty} \frac{r_k(b)}{q_k} - 2^{-\lfloor \frac{m-1}{2} \rfloor}\right\} = 1$$

so $\underline{R}(x) = \mathbf{r}_0$ almost everywhere. The proof for $\overline{R}$ is similar using the points c or d instead of b. □

Corrolary 9. *Given an irrational $\alpha = [0, a_1, a_2, \ldots]$ with convergents q_k, $\gamma = \frac{\sqrt{5}+1}{2}$, set*

$$\mathbf{r}_0 = \liminf_{k\to\infty} \frac{q_k}{q_{k+1} + q_k - 1}, \quad \mathbf{r}_1 = \limsup_{k\to\infty} \frac{q_{k+1}}{q_k}$$

Then $\mathbf{r}_0 \leq \underline{R}(x) \leq \overline{R}(x) \leq \mathbf{r}_1$ *for every* $x \in \Sigma_\alpha$ *and* $\underline{R}(x) = \mathbf{r}_0$, $\overline{R}(x) = \mathbf{r}_1$ *almost everywhere.*

If $\{a_k\}_{k=0}^{\infty}$ *is unbounded, then* $\mathbf{r}_0 = 0$ *and* $\mathbf{r}_1 = \infty$. *In the case of bounded* $\{a_k\}_{k=0}^{\infty}$,

$$\frac{1}{M+2} \leq \mathbf{r}_0 \leq \gamma^{-2} < \gamma \leq \mathbf{r}_1 \leq M+1$$

where $M = \limsup a_k$. *Moreover,* $\mathbf{r}_1 = \gamma$*(resp.* $\mathbf{r}_0 = \gamma^{-2}$*) if and only if* $M = 1$.

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